Log Hochschild homology via the log diagonal

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Outline

The module of Kähler differentials

Let M be a real smooth manifold and let $p \in M$.
The *tangent space* $T_p(M)$ is the collection of *derivations in* p:

$$T_p(M) := \{D\colon C^\infty(M) \to \mathbb{R} \mid D(fg) = D(f)g(p) + f(p)D(g)\}.$$

This generalizes: Let A be a commutative algebra (over R) and J an A-module:

$$\mathrm{Der}_R(A, J) := \{d\colon A \to J \mid d(ab) = ad(b) + bd(a)\}.$$

We recover $T_p(M)$ by setting $R = \mathbb{R}$, $A = C^\infty(M)$, and $J = \mathbb{R}$ (module structure by evaluation at p).

Definition

The A-module $\Omega^1_{A|R}$ is the corepresenting object for derivations:

$$\mathrm{Der}_R(A, J) \cong \mathrm{Hom}_{\mathrm{Mod}_A}(\Omega^1_{A|R}, J).$$

One concrete description is $\Omega^1_{A|R} \cong (\bigoplus_{a\in A} A\{da\})/\sim$.

Example

- $R = \mathbb{Z}$, $A = \mathbb{Z}[x] \implies \Omega^1_{A|R} = A\{dx\}$.
- $R = \mathbb{Z}$, $A = \mathbb{Z}[x, y]/(xy) \implies \Omega^1_{A|R} = A\{dx, dy\}/(ydx + xdy)$.
- $R = \mathbb{R}$, $A = C^\infty(M) \implies \mathbb{R} \otimes_{C^\infty(M)} \Omega^1_{C^\infty(M)|\mathbb{R}} \cong T^*_p(M)$.

Another very useful description of $\Omega^1_{A|R}$ is

$$\Omega^1_{A|R} \cong I/I^2, \quad I := \ker(A \otimes_R A \to A).$$

In algebro-geometric terms, this is the conormal of the diagonal

$$\mathrm{Spec}(A) \to \mathrm{Spec}(A \otimes_R A).$$

In general, is is always true that

$$\mathrm{Tor}_1^S(S/I, S/J) \cong (I \cap J)/IJ.$$

It follows that $\mathrm{Tor}_1^{A \otimes_R A}(A, A) \cong I/I^2 \cong \Omega^1_{A|R}$.
Two graded commutative rings associated to $\Omega^1_{A|R}$:

- The exterior algebra $\Omega^*_{A|R} = \Lambda^*_A \Omega^1_{A|R}$.
- The graded commutative ring $\mathrm{Tor}_*^{A \otimes_R A}(A, A)$.

Theorem (Hochschild–Kostant–Rosenberg, 1962)

If $R \to A$ is smooth, the canonical map

$$\Omega^*_{A|R} \to \mathrm{Tor}_*^{A \otimes_R A}(A, A)$$

is an isomorphism of graded commutative rings.

Hochschild homology

By definition, the Tor-groups $\mathrm{Tor}_*^{A\otimes_R A}(A, A)$ are the homology groups of the derived tensor product $A \otimes^{\mathbb{L}}_{A\otimes_R A} A$.

- If A is flat over R, this can be computed via the *cyclic bar construction* $B^{\mathrm{cy}}_R(A)$:

$$B^{\mathrm{cy}}_R(A) = \cdots \xrightarrow{\partial_3} A^{\otimes_R 3} \xrightarrow{\partial_2} A^{\otimes_R 2} \xrightarrow{\partial_1} A \xrightarrow{\partial_0} 0,$$

  where

$$\partial_q(a_0 \otimes \cdots \otimes a_q) = \Sigma_{i=0}^{q-1}(-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q \\ +(-1)^q a_q a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1}$$

- By definition, the homology groups of the cyclic bar construction $H_n B^{\mathrm{cy}}_R(A)$ are the *Hochschild homology* groups $\mathrm{HH}^R_n(A)$.

Example

Let $R = \mathbb{Z}$ and $A = \mathbb{Z}[x]$. To compute $\mathrm{HH}^{\mathbb{Z}}_n(\mathbb{Z}[x])$, we can compute the Tor-groups $\mathrm{Tor}_n^{\mathbb{Z}[x]\otimes_{\mathbb{Z}}\mathbb{Z}[x]}(\mathbb{Z}[x], \mathbb{Z}[x])$. Write $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x] \cong \mathbb{Z}[y, z]$, and replace $\mathbb{Z}[x]$ with the chain complex

$$\cdots \to 0 \to \mathbb{Z}[y, z] \xrightarrow{\cdot(y-z)} \mathbb{Z}[y, z] \to 0.$$

Applying $\mathbb{Z}[x] \otimes_{\mathbb{Z}[y,z]} -$ to this chain complex, we obtain $\cdots \to 0 \to \mathbb{Z}[x] \xrightarrow{0} \mathbb{Z}[x] \to 0$.

Derived Hochschild homology

To sketch a proof of the HKR-theorem, we shall make use of a derived variant of Hochschild homology:

Definition

Let A be a (not necessarily flat) commutative R-algebra. The *derived Hochschild* (or *Shukla*) homology of A is the derived self-intersections of the diagonal:

$$\mathrm{HH}^R(A) := A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_R A} A.$$

This can be computed by $\mathrm{diag}(B^{\mathrm{cy}}_R(A_\bullet))$ for $A_\bullet \xrightarrow{\simeq} A$ a *polynomial* R-algebra resolution. From the definition, we obtain homotopy pushout squares

$$\begin{array}{ccc} A \otimes^{\mathbb{L}}_R A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{HH}^R(A) \end{array} \quad \xrightarrow{I/I^2} \quad \begin{array}{ccc} \mathbb{L}_{A|R} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{A|R}[1]. \end{array}$$

Here $\mathbb{L}_{A|R}$ is the *cotangent complex* of A relative to R, often described by the simplicial object $A \otimes_{A_\bullet} \Omega^1_{A_\bullet|R}$

Sketch proof of HKR.

It is true for $R[x]$, hence it is true for $R[x_1, \dots, x_n]$. Hence

$$\pi_n B^{\mathrm{cy}}_R(A_\bullet) \cong \Omega^n_{A_\bullet|R} \cong A \otimes_{A_\bullet} \Omega^n_{A_\bullet|R} = \Lambda^n_A \mathbb{L}_{A|R}.$$

But if $R \to A$ is smooth, $B^{\mathrm{cy}}_R(A_\bullet) \xrightarrow{\simeq} B^{\mathrm{cy}}_R(A)$ and $\mathbb{L}_{A|R} \simeq \Omega^1_{A|R}[0]$. □

One consequence of the Hochschild–Kostant–Rosenberg:

Corollary

If $R \to A$ is étale, then the unit map $A \to \mathrm{HH}^R(A)$ is an equivalence. □

Examples of étale maps include $R \to R[x^{-1}]$, $\mathbb{Z} \to \mathbb{Z}[1/2, i], \dots$. A closely related statement is:

Theorem (Weibel–Geller)

If $A \to B$ is an étale map of commutative R-algebras, the canonical map

$$B \otimes_A \mathrm{HH}^R(A) \to \mathrm{HH}^R(B)$$

is an equivalence.

The main goal of this talk is to generalize these results using *log geometry*:

- Get a version of HKR for $\mathbb{Z}[x, y]/(xy)$, despite this not being smooth.
- Get a version of étale base-change for $\mathbb{Z}_{(3)} \to \mathbb{Z}_{(3)}[\sqrt{3}]$, despite this not being étale.

Logarithmic rings

Definition (Kato, 1989)

A *pre-logarithmic ring* (R, P, β) consists of

- a commutative monoid P;
- a commutative ring R;
- a map $\beta\colon P \to (R, \cdot)$; equivalently, a map $\bar{\beta}\colon \mathbb{Z}[P] \to R$.

Example

Any ring R admits a *trivial pre-log structure* $(R, \{1\})$.

Example

$(\mathbb{Z}_{(3)}, \langle 3 \rangle)$ is a pre-log ring, with

$$\langle 3 \rangle = \{3^0, 3^1, 3^2, \dots\}$$

and structure map the inclusion.

Example

$(\mathbb{Z}[x_1, \dots, x_n], \langle x_1, \dots, x_k \rangle)$, where $\langle x_1, \dots, x_k \rangle = \langle x_1 \rangle \oplus \cdots \oplus \langle x_k \rangle$ is the free commutative monoid of rank $k \leq n$.

Logarithmic rings

Definition

A *logarithmic ring* (R, P, β) is a pre-log ring with the property that $\widetilde{\beta}$ in the pullback

$$\begin{array}{ccc} \beta^{-1}\mathrm{GL}_1(R) & \xrightarrow{\widetilde{\beta}} & \mathrm{GL}_1(R) \\ \downarrow & & \downarrow \\ P & \xrightarrow{\beta} & (R, \cdot) \end{array}$$

is an isomorphism.

Example

Trivial log ring $(R, \mathrm{GL}_1(R))$.

Example

$(\mathbb{Z}_{(3)}, \langle 3 \rangle \times \mathrm{GL}_1(\mathbb{Z}_{(3)}))$ is a log ring, with structure map the inclusion.

There is a factorization

$$(\mathbb{Z}_{(3)}, \mathrm{GL}_1(\mathbb{Z}_{(3)})) \to (\mathbb{Z}_{(3)}, \langle 3 \rangle \times \mathrm{GL}_1(\mathbb{Z}_{(3)})) \to (\mathbb{Q}, \mathrm{GL}_1(\mathbb{Q}))$$

in the category of log rings. More generally, for any log ring (R, P) there is a factorization

$$(R, \mathrm{GL}_1(R)) \to (R, P) \to (R[P^{-1}], \mathrm{GL}_1(R[P^{-1}])),$$

where $R[P^{-1}] := R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{gp}}]$.

Logarithmic derivations

Let $(f, f^\flat)\colon (R, P, \beta) \to (A, M, \alpha)$ be a map of pre-log rings.

Definition

A *logarithmic derivation* $(d, d^\flat) \in \mathrm{Der}_{(R,P)}((A, M), J)$ consists of

- $d \in \mathrm{Der}_R(A, J)$ such that $d(\alpha(m)) = \alpha(m)d^\flat(m)$
- $d^\flat\colon M \to (J, +)$ $d^\flat(f^\flat(p)) = 0.$

Suppose that $P = \{1\}$, so that part of the definition is simply a map of commutative monoids $d^\flat\colon M \to (J, +)$. We notice that

$$\mathrm{Hom}_{\mathrm{CMon}}(M, (J, +)) \cong \mathrm{Hom}_{\mathrm{Ab}}(M^{\mathrm{gp}}, (J, +)) \cong \mathrm{Hom}_{\mathrm{Mod}_A}(A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}, J).$$

Fact

$\mathrm{Der}_{(R,P)}((A, M), J)$ *is corepresented by*

$$\Omega^1_{(A,M)|(R,P)} := \frac{\Omega^1_{A|R} \oplus (A \otimes_{\mathbb{Z}} (M^{\mathrm{gp}}/P^{\mathrm{gp}}))}{d\alpha(m) = \alpha(m) \otimes [m]}$$

Logarithmic derivations

There is a notion of logarithmic derivations $\mathrm{Der}_{(R,P)}((A,M),J)$, and this is corepresented by

$$\Omega^1_{(A,M)|(R,P)} := \frac{\Omega^1_{A|R} \oplus (A \otimes_{\mathbb{Z}} (M^{\mathrm{gp}}/P^{\mathrm{gp}}))}{d\alpha(m) = \alpha(m) \otimes [m]}.$$

We write $\mathrm{dlog}(m)$ for $1 \otimes [m]$, so that the defining relation reads $d\alpha(m) = \alpha(m)\mathrm{dlog}(m)$.

Example

There is a map of pre-log rings $(\mathbb{Z}_{(3)}, \langle 3 \rangle) \to (\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$. For this map we have $\Omega^1_{(A,M)|(R,P)} \cong 0$, despite $\Omega^1_{A|R}$ being non-trivial: Indeed, $\langle \sqrt{3} \rangle^{\mathrm{gp}}/\langle 3 \rangle^{\mathrm{gp}}$ is cyclic of order two.

Example

There is a short exact sequence

$$0 \to \Omega^1_{\mathbb{Z}[x]} \to \Omega^1_{(\mathbb{Z}[x], \langle x \rangle)} \to \mathbb{Z} \to 0$$

isomorphic to $0 \to \mathbb{Z}[x]\{dx\} \to \mathbb{Z}[x]\{\mathrm{dlog}(x)\} \to \mathbb{Z} \to 0$ sending dx to $dx = x\mathrm{dlog}(x)$. Using base-change and transitivity properties of the (log) Kähler differentials, this gives rise to sequences

$$0 \to \Omega^1_A \to \Omega^1_{(A, \langle a \rangle)} \to A/(a) \to 0.$$

Repletion

Goal: Describe $\Omega^1_{(A,M)|(R,P)} \cong I/I^2$ for some $I := \ker(B \to A)$. (For $B = A \otimes_R A$, this is precisely the Kähler differentials $\Omega^1_{A|R}$.)

Definition (Kato–Saito, Rognes)

$K \to M$ map of commutative monoids such that $K^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is surjective. The *repletion* $K^{\mathrm{rep}} \to M$ by the pullback square

$$\begin{array}{ccc} K^{\mathrm{rep}} & \longrightarrow & K^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{\mathrm{gp}}. \end{array}$$

The universal property of the pullback induces a map $K \to K^{\mathrm{rep}}$.

In the same way that the map $A \otimes_R A \to A$ is relevant for describing the Kähler differentials, might suspect that the map $M \oplus_P M \to M$ is relevant for the log Kähler differentials.

- Consider the addition map $+\colon \mathbb{N} \oplus \mathbb{N} \to \mathbb{N}$. In this case $(\mathbb{N} \oplus \mathbb{N})^{\mathrm{rep}} \to \mathbb{N}$ fits in a pullback square

$$\begin{array}{ccc} (\mathbb{N} \oplus \mathbb{N})^{\mathrm{rep}} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{N} & \longrightarrow & \mathbb{Z}. \end{array}$$

  Isomorphic to $\mathbb{N} \oplus \mathbb{Z}$. The map from $\mathbb{N} \oplus \mathbb{N}$ sends $(n, m) \mapsto (n + m, m)$.
- More generally, we have $(M \oplus_P M)^{\mathrm{rep}} \cong M \oplus (M^{\mathrm{gp}}/P^{\mathrm{gp}})$.

We described $\Omega^1_{A|R}$ as the indecomposables of the multiplication map $A \otimes_R A \to A$; the conormal of the diagonal. For $(R, P) \to (A, M)$, there is a pre-log ring $(A \otimes_R A, M \oplus_P M)$ with adjoint structure map

$$\mathbb{Z}[M \oplus_P M] \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M] \to A \otimes_R A.$$

We are led to think of the multiplication map $(A \otimes_R A, M \oplus_P M) \to (A, M)$.

Definition

The *replete base-change* $((A \otimes_R A)^{\mathrm{rep}}, (M \oplus_P M)^{\mathrm{rep}})$ is given by

$$((A \otimes_R A) \otimes_{\mathbb{Z}[M \oplus_P M]} \mathbb{Z}[(M \oplus_P M)^{\mathrm{rep}}], (M \oplus_P M)^{\mathrm{rep}}) \to (A, M).$$

Proposition (Kato–Saito, 2004)

The indecomposables of $(A \otimes_R A)^{\mathrm{rep}} \to A$ *is* $\Omega^1_{(A,M)|(R,P)}$.

In algebro-geometric terminology, $\Omega^1_{(A,M)|(R,P)}$ is the conormal of the *log diagonal*.

Logarithmic Hochschild homology

If $P \to M$ is a map of commutative monoids, one can form the *cyclic bar construction* $B^{\mathrm{cy}}_P(M)$. It has q-simplices $B^{\mathrm{cy}}_P(M)_q = M^{\oplus_P(1+q)}$, and $\mathbb{Z}[B^{\mathrm{cy}}_P(M)] \cong B^{\mathrm{cy}}_{\mathbb{Z}[P]}(\mathbb{Z}[M])$.

Definition (Rognes)

The *replete bar construction* $B^{\mathrm{rep}}_P(M)$ is the (levelwise) repletion of the (levelwise) multiplication map $B^{\mathrm{cy}}_P(M) \to M$.

- By definition, the replete bar construction fits in a pullback diagram

$$\begin{array}{ccc} B^{\mathrm{rep}}_P(M) & \longrightarrow & B^{\mathrm{cy}}_{P^{\mathrm{gp}}}(M^{\mathrm{gp}}) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{\mathrm{gp}}. \end{array}$$

  of simplicial commutative monoids.
- There is an isomorphism $B^{\mathrm{rep}}_P(M) \cong M \oplus B(M^{\mathrm{gp}}/P^{\mathrm{gp}})$, so that the q-simplices of the replete bar construction are $M \oplus (M^{\mathrm{gp}}/P^{\mathrm{gp}})^{\oplus q}$.

Definition (Rognes)

The *log Hochschild homology* $\mathrm{HH}^{(R,P)}(A, M)$ is the pushout of the diagram

$$\mathrm{HH}^R(A) \leftarrow \mathrm{HH}^{\mathbb{Z}[P]}(\mathbb{Z}[M]) \cong \mathbb{Z}[B^{\mathrm{cy}}_P(M)] \to \mathbb{Z}[B^{\mathrm{rep}}_P(M)].$$

Log Hochschild homology is the pushout of

$$\mathrm{HH}^R(A) \leftarrow \mathrm{HH}^{\mathbb{Z}[P]}(\mathbb{Z}[M]) \cong \mathbb{Z}[B_P^{\mathrm{cy}}(M)] \to \mathbb{Z}[B_P^{\mathrm{rep}}(M)].$$

Example (Rognes–Sagave–Schlichtkrull)

Recall that there are short exact sequences

$$0 \to \Omega^1_A \to \Omega^1_{(A,\langle a \rangle)} \to A/(a) \to 0$$

for $a \in A$ a non-zero divisor. Similarly, there are homotopy cofiber sequences

$$\mathrm{HH}(A) \to \mathrm{HH}(A, \langle a \rangle) \to \mathrm{HH}(A/(a))[1]$$

which recovers the former sequence on π_1.

Our goal now is to give a description of log Hochschild homology analogous to

$$\mathrm{HH}^R(A) \simeq A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_R A} A.$$

Theorem (L.)

Log Hochschild homology is the derived self-intersections of the log diagonal:

$$\mathrm{HH}^{(R,P)}(A, M) \simeq A \otimes^{\mathbb{L}}_{(A \otimes^{\mathbb{L}}_R A)^{\mathrm{rep}}} A.$$

So we can compute log Hochschild homology as the derived self-intersections of the log diagonal

$$\mathrm{HH}^{(R,P)}(A,M) \simeq A \otimes^{\mathbb{L}}_{(A\otimes^{\mathbb{L}}_R A)^{\mathrm{rep}}} A.$$

Example

Consider $(A, M) = (\mathbb{Z}[x], \langle x \rangle)$. Then $(\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x])^{\mathrm{rep}} \cong \mathbb{Z}[y, z^{\pm 1}]$. To compute log Hochschild homology, use the resolution

$$\cdots \to 0 \to \mathbb{Z}[y, z^{\pm 1}] \xrightarrow{(1-z)} \mathbb{Z}[y, z^{\pm 1}] \to 0$$

of $\mathbb{Z}[x]$. Applying $\mathbb{Z}[x] \otimes_{\mathbb{Z}[y,z^{\pm 1}]} -$, obtain the complex

$$\cdots \to 0 \to \mathbb{Z}[x] \xrightarrow{0} \mathbb{Z}[x] \to 0.$$

While the (log) Hochschild homologies $\mathrm{HH}^{\mathbb{Z}}(\mathbb{Z}[x])$ and $\mathrm{HH}^{(\mathbb{Z},\{1\})}(\mathbb{Z}[x], \langle x \rangle)$ have isomorphic homology groups, the map

$$\mathbb{Z}[x]\{dx\} \to \mathbb{Z}[x]\{\mathrm{dlog}(x)\}$$

is not an isomorphism.

Definition (Slightly informal)

We say that $(R, P) \to (A, M)$ is

- *log étale* if $R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M] \to A$ is étale and $|M^{\mathrm{gp}}/P^{\mathrm{gp}}| \in \mathrm{GL}_1(A)$.
- *log smooth* if $R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M] \to A$ is smooth and $M^{\mathrm{gp}}/P^{\mathrm{gp}}$ finitely generated with torsion of order invertible in A.

Example

$(\mathbb{Z}_{(3)}, \langle 3 \rangle) \to (\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$ is log étale but not étale.

Example

Weird but important pre-log structure: $\langle t \rangle \to (\mathbb{Z}, \cdot)$ sending $t^0 \mapsto 1$, $t^n \mapsto 0$. Adjoint to $\mathbb{Z}[t] \to \mathbb{Z}$ sending t to 0. Map of pre-log rings

$$(\mathbb{Z}, \langle t \rangle) \xrightarrow{(i,\Delta)} (\mathbb{Z}[x, y]/(xy), \langle x \rangle \oplus \langle y \rangle).$$

Log smooth but not smooth.

The log HKR-theorem

Theorem (Binda–Park–L.–Østvær)

Let $(R,P) \to (A,M)$ be log smooth and assume further that $\mathbb{Z}[P] \to \mathbb{Z}[M]$ is flat. Then the canonical map

$$\Omega^*_{(A,M)|(R,P)} \to \pi_* \mathrm{HH}^{(R,P)}(A,M)$$

is an isomorphism.

We want to use the same proof strategy as in the ordinary case. By definition, there are homotopy pushout squares

$$\begin{array}{ccc} (A \otimes^{\mathbb{L}}_R A)^{\mathrm{rep}} & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{HH}^{(R,P)}(A,M) \end{array} \quad \xrightarrow{I/I^2} \quad \begin{array}{ccc} \mathbb{L}_{(A,M)|(R,P)} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{(A,M)|(R,P)}[1]. \end{array}$$

Here $\mathbb{L}_{(A,M)|(R,P)}$ is the *log cotangent complex*, often defined as $A \otimes_{A_\bullet} \Omega^1_{(A_\bullet, M_\bullet)|(R,P)}$.

- $(A_\bullet, M_\bullet) \to (A,M)$ is a resolution by free (R,P)-algebras

$$(R[x_1, \dots, x_n, y_1, \dots, y_m], P \oplus \langle x_1, \dots, x_n \rangle)$$

- If $(R,P) \to (A,M)$ log étale and $\mathbb{Z}[P] \to \mathbb{Z}[M]$ is flat, $\mathbb{L}_{(A,M)|(R,P)} \simeq *$.
- If $(R,P) \to (A,M)$ log smooth and $\mathbb{Z}[P] \to \mathbb{Z}[M]$ is flat, $\mathbb{L}_{(A,M)|(R,P)} \simeq \Omega^1_{(A,M)|(R,P)}$.

Recall that $(\mathbb{Z}_{(3)}, \langle 3 \rangle) \to (\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$ is log étale. We obtain a base-change formula

$$\mathbb{Z}_{(3)}[\sqrt{3}] \otimes_{\mathbb{Z}_{(3)}} \mathrm{HH}(\mathbb{Z}_{(3)}, \langle 3 \rangle) \xrightarrow{\simeq} \mathrm{HH}(\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle).$$

More generally, we have

Theorem (L.)

Let $R \to A$ be a tamely ramified extension of discrete valuation rings in mixed characteristic with perfect residue fields. Then the canonical map

$$A \otimes_R \mathrm{HH}(R, \langle \pi_R \rangle) \to \mathrm{HH}(A, \langle \pi_A \rangle)$$

is an equivalence.

Let $R \to A$ be a map of commutative ring spectra. Define $\mathrm{THH}^R(A) := B^{\mathrm{cy}}_R(A)$ (implicitly derived). As before there is a homotopy pushout

$$\begin{array}{ccc} A \otimes_R A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{THH}^R(A) \end{array} \quad \xrightarrow{I/I^2} \quad \begin{array}{ccc} \mathbb{L}_{A|R} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{A|R}[1]. \end{array}$$

Here $\mathbb{L}_{A|R}$ is the *topological André–Quillen homology* of A relative to R. This is often denoted by $\mathrm{TAQ}^R(A)$.

Example

- $L_p \to \mathrm{KU}_p$ inclusion of Adams summand. Satisfies base-change

$$\mathrm{KU}_p \otimes_{L_p} \mathrm{THH}(L_p) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{KU}_p)$$

  on THH. Not true for $\ell_p \to \mathrm{ku}_p$.
- $\mathrm{KO} \to \mathrm{KU}$ complexification map. Satisfies base-change

$$\mathrm{KU} \otimes_{\mathrm{KO}} \mathrm{THH}(\mathrm{KO}) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{KU})$$

  on THH. Not true for $\mathrm{ko} \to \mathrm{ku}$.

Definition (First guess)

A *pre-log ring spectrum* (R, P, β) consists of

- a commutative ring spectrum R;
- an $\mathbb{E}_\infty$-space P;
- a map $P \to \Omega^\infty(R)$; equivalently a map $\mathbb{S}[P] := \Sigma^\infty_+ P \to R$.

Problem: This notion will not give rise to pre-log structures $(R, \langle x \rangle)$ satisfying $R[\langle x \rangle^{-1}] \simeq R[x^{-1}]$ for x a homotopy class in strictly positive degree.

Example

We would like a pre-log ring spectrum $(\mathrm{ku}, \langle u \rangle)$ with $\mathrm{ku}[\langle u \rangle^{-1}] \simeq \mathrm{KU}$, where $u \in \pi_2(\mathrm{ku})$ is the Bott class. This does not exist with the above definition (Rognes).

Solution (Sagave–Schlichtkrull): Consider instead $\mathbb{E}_\infty$-spaces with a $QS^0 := \Omega^\infty \Sigma^\infty S^0$-grading. This comes with a $(\mathbb{S}_*, \Omega^\infty_*)$-adjunction, a good theory of group completion, repletion...

Definition

A pre-log ring spectrum (R, P, β) consists of

- a commutative ring spectrum R;
- a QS^0-graded $\mathbb{E}_\infty$-space P;
- a map $P \to \Omega^\infty_*(R)$; equivalently a map $\mathbb{S}_*[P] \to R$.

All examples are formed in the framework of Sagave–Schlichtkrull (2012) and Sagave (2016).

- If (R, P) is an ordinary pre-log ring, there is a pre-log ring spectrum (HR, FP). It has adjoint structure map

$$\mathbb{S}_*[FP] \cong \mathbb{S}[P] \to H\mathbb{Z}[P] \to HR.$$

- There is a pre-log ring spectrum $(\mathrm{ku}, \langle u \rangle_*)$ such that

$$\mathrm{ku} \otimes_{\mathbb{S}_*[\langle u \rangle_*]} \mathbb{S}_*[\langle u \rangle_*^{\mathrm{gp}}] \simeq \mathrm{KU}.$$

- Similarly for $(\ell_p, \langle v_1 \rangle_*)$, with $v_1 \in \pi_{2(p-1)}(\ell_p)$. There is a map

$$(\ell_p, \langle v_1 \rangle_*) \to (\mathrm{ku}_p, \langle u \rangle_*).$$

- Similarly, $(\mathrm{ko}, \langle \beta \rangle_*) \to (\mathrm{ku}, \langle u \rangle_*)$.

If $(R, P) \to (A, M)$ is a map of pre-log ring spectra, $\mathrm{THH}^R(A)$ participates in a pre-log structure $B_P^{\mathrm{cy}}(M) \to \Omega_*^\infty(\mathrm{THH}^R(A))$ adjoint to

$$\mathbb{S}_*[B_P^{\mathrm{cy}}(M)] \cong B^{\mathrm{cy}}_{\mathbb{S}_*[P]}(\mathbb{S}_*[M]) \to B_R^{\mathrm{cy}}(A) =: \mathrm{THH}^R(A).$$

Definition (Rognes–Sagave–Schlichtkrull)

The *log topological Hochschild homology* $\mathrm{THH}^{(R,P)}(A, M)$ is the homotopy pushout of

$$\mathrm{THH}^R(A) \leftarrow \mathrm{THH}^{\mathbb{S}_*[P]}(\mathbb{S}_*[M]) \cong \mathbb{S}_*[B_P^{\mathrm{cy}}(M)] \to \mathbb{S}_*[B_P^{\mathrm{cy}}(M)^{\mathrm{rep}}],$$

where the repletion is taken with respect to $B_P^{\mathrm{cy}}(M) \to M$.

Recall that there are homotopy cofiber sequences

$$\mathrm{HH}(A) \to \mathrm{HH}(A, \langle a \rangle) \to \mathrm{HH}(A/(a))[1]$$

in ordinary log Hochschild homology.

Theorem (Rognes–Sagave–Schlichtkrull)

There is a homotopy cofiber sequence

$$\mathrm{THH}(\mathrm{ku}) \to \mathrm{THH}(\mathrm{ku}, \langle u \rangle_*) \to \Sigma\mathrm{THH}(\mathbb{Z}).$$

It is tempting to think of the homotopy cofiber sequence

$$\mathrm{THH}(\mathrm{ku}) \to \mathrm{THH}(\mathrm{ku}, \langle u \rangle_*) \to \Sigma\mathrm{THH}(\mathbb{Z})$$

as an analogue of Blumberg–Mandell's

$$K(\mathbb{Z}) \to K(\mathrm{ku}) \to K(\mathrm{KU}).$$

This is not so clear-cut: for ko, there is a homotopy cofiber sequence

$$\mathrm{THH}(\mathrm{ko}) \to \mathrm{THH}(\mathrm{ko}, \langle w \rangle_*) \to \Sigma\mathrm{THH}(\tau_{\leq 7}(\mathrm{ko}))$$

in log THH. Blumberg–Mandell's argument gives (Barwick–Lawson) a homotopy cofiber sequence

$$K(\mathbb{Z}) \to K(\mathrm{ko}) \to K(\mathrm{KO}).$$

- If not from localization properties analogous to those in algebraic K-theory, where do the sequences in log THH "come from?"

A version $\mathbb{L}_{(A,M)|(R,P)}$ of log TAQ is studied by Rognes–Sagave. From the definitions it is not clear that this relates to log THH in the manner we would expect from the linear case.

Theorem (L.)

This all works out: There is an equivalence

$$\mathbb{L}_{(A,M)|(R,P)} \simeq I/I^2, \quad I := \mathrm{fib}((A \otimes_R A)^{\mathrm{rep}} \to A),$$

and $\mathrm{THH}^{(R,P)}(A, M) \simeq A \otimes_{(A \otimes_R A)^{\mathrm{rep}}} A$.

More precisely, there is a homotopy pushout

$$\begin{array}{ccc} (A \otimes_R A)^{\mathrm{rep}} & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{THH}^{(R,P)}(A, M) \end{array} \quad \xrightarrow{I/I^2} \quad \begin{array}{ccc} \mathbb{L}_{(A,M)|(R,P)} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{(A,M)|(R,P)}[1] \end{array}$$

Moreover, $\mathbb{L}_{(A,M)|(R,P)}$ agrees with previous notions of log TAQ.

Definition (Slightly informal)

A map $(R, P) \to (A, M)$ of pre-log ring spectra is

- *log étale* if $R \otimes_{\mathbb{S}_*[P]} \mathbb{S}_*[M] \to A$ is étale and $A \otimes (M^{\mathrm{gp}}/P^{\mathrm{gp}}) \simeq *$;
- *log smooth*? (Krause–Nikolaus' forthcoming spherical HKR-theorem will provide some hints.)

Let $\ell_p \to \mathrm{ku}_p$ denote the inclusion of the p-complete Adams summand. On coefficient rings, $\mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u]$ sends v_1 to u^{p-1}.

Theorem (Sagave, reformulated)

The map $(\ell_p, \langle v_1 \rangle_*) \to (\mathrm{ku}_p, \langle u \rangle_*)$ *of pre-log ring spectra is log étale.*

As a consequence, the log cotangent complex is contractible.

Theorem (L.)

- $(R, P) \to (A, M)$ *log étale of discrete pre-log rings induces log étale map of pre-log ring spectra provided that* $\mathbb{Z}[P] \to \mathbb{Z}[M]$ *is flat.*
- $(R, P) \to (A, M)$ *log étale of pre-log ring spectra implies that*

$$A \otimes_R \mathrm{THH}(R, P) \to \mathrm{THH}(A, M)$$

is an equivalence.

We get examples of base-change in log THH:

- $\mathbb{Z}_{(3)}[\sqrt{3}] \otimes_{\mathbb{Z}_{(3)}} \mathrm{THH}(\mathbb{Z}_{(3)}, \langle 3 \rangle) \xrightarrow{\simeq} \mathrm{THH}(\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$
- $\mathrm{ku}_p \otimes_{\ell_p} \mathrm{THH}(\ell_p, \langle v_1 \rangle_*) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{ku}_p, \langle u \rangle_*)$.

The latter recovers a theorem by Rognes–Sagave–Schlichtkrull. Heuristically, this exhibits the inclusion of the Adams summand as a tamely ramified extension.

- The complexification map $\mathrm{ko} \to \mathrm{ku}$ participates in a map

$$(\mathrm{ko}, \langle \beta \rangle_*) \to (\mathrm{ku}, \langle u \rangle_*)$$

  of pre-log ring spectra.

Theorem (Höning–Richter)

The log cotangent complex $\mathbb{L}_{(\mathrm{ku}, \langle u \rangle_*)|(\mathrm{ko}, \langle \beta \rangle_*)}$ is not contractible.

This implies that $(\mathrm{ko}, \langle \beta \rangle_*) \to (\mathrm{ku}, \langle u \rangle_*)$ is not log étale. Heuristically, this means that the complexification map if wildly ramified.

Future work: Residue sequences in log THH

With the close relationship between log THH and log geometry at hand, we revisit the question

- If not from localization properties analogous to those in algebraic K-theory, where do the sequences in log THH "come from?"

One answer: from residue sequences in logarithmic geometry. The arguments of Blumberg–Mandell give rise to a localization sequence

$$K(\mathbb{Z}_p) \to K(\ell_p) \to K(L_p),$$

which can be rewritten as

$$K(\mathbb{Z}_p) \to K(BP\langle 1\rangle) \to K(E(1)).$$

Theorem (Antieau–Barthel–Gepner)

There are no localization sequences

$$K(BP\langle n-1\rangle) \to K(BP\langle n\rangle) \to K(E(n))$$

for $n \geq 2$.

Rognes–Sagave–Schlichtkrull's setup is only defined for $\mathbb{E}_\infty$-structures. Using recent work of Hahn–Wilson, we can construct "good" pre-log structures on $BP\langle n\rangle$, using periodic complex cobordism MUP.

Future work: Residue sequences in log THH

This gives rise to localization sequences

$$\mathrm{THH}(BP\langle n\rangle) \to \mathrm{THH}(BP\langle n\rangle, \langle v_n\rangle_*) \to \Sigma\mathrm{THH}(BP\langle n-1\rangle).$$

Now worthwhile to ask the opposite question:

- Are there residue sequences in (variants of) algebraic K-theory that match the residue sequences in THH under a trace map?

We hope that versions of cyclic K-theory will realize such a program. Some other questions:

- Stronger HKR-statements in characteristic zero (Toën–Vezzosi for ordinary HH).
- Ramification at chromatic primes? (Example: $\mathrm{ku}_p \to \mathrm{THH}^{\ell_p}(\mathrm{ku}_p)$ a $K(1)$-local equivalence [Angeltveit].)
- Log deformation theory and tamely ramified extensions.
- Bhatt–Morrow–Scholze style filtration on log TC. (Related to forthcoming work of Bhatt–Clausen–Mathew).

Thanks for listening!