On the relationship between logarithmic TAQ and logarithmic THH

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$$d_i(m_0, \dots, m_q) = \begin{cases} (m_0, \dots, m_i m_{i+1}, \dots, m_q), & \text{if } 0 \leq i < q, \\ (m_q m_0, \cdots, m_{q-1}), & \text{if } i = q \end{cases}$$

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and degeneracies $s_j(m_0, \dots, m_q) = (m_0, \dots, m_{j-1}, 1, m_j, \dots, m_q)$.

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Lemma

$$B^{\mathrm{cy}}_P(M)_\bullet \cong P \boxtimes_{B^{\mathrm{cy}}_1(P)_\bullet} B^{\mathrm{cy}}_1(M)_\bullet = P \boxtimes_{B^{\mathrm{cy}}(P)_\bullet} B^{\mathrm{cy}}(M)_\bullet.$$

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Definition

Let $X_\bullet$ be a finite simplicial set.

- Let $X_\bullet \otimes_P M$ be the simplicial commutative monoid $[q] \mapsto M^{\boxtimes_P |X_q|} = M \boxtimes_P \cdots \boxtimes_P M$, the $|X_q|$-fold coproduct.
- Assume $X_\bullet$ is pointed and let $M \to N \to M$ be an object of $\mathcal{CM}_{M//M}$. Let $X_\bullet \odot_M N = \mathrm{colim}(M \leftarrow N \to X_\bullet \otimes_M N)$.

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Lemma

Fix $P \to M$. Then $B^{\mathrm{cy}}_P(M)_\bullet \cong S^1_\bullet \otimes_P M \cong S^1_\bullet \odot_M (M \boxtimes_P M)$.

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The tensors now participate in simplicial model structures. In particular, $S^1 \odot_A (A \wedge_R A)$ models the suspension of $A \wedge_R A$ in $\mathcal{C}\mathrm{Sp}^\Sigma_{A//A}$.

Topological André–Quillen homology

Let $R \to A$ be a map of discrete commutative rings. There is an A-module $\Omega^1_{A|R}$ such that

$$\mathrm{Mod}_A(\Omega^1_{A|R}, M) \cong \mathrm{Der}_R(A, M) = \mathrm{CRing}_{R//A}(A, A \oplus M).$$

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Theorem (Basterra–Mandell)

There are Quillen adjunctions

$$\mathrm{Mod}_A \underset{\longrightarrow}{\overset{Q_A}{\longleftarrow}} \mathrm{Nuca}_A \underset{I_A}{\overset{\longrightarrow}{\longleftarrow}} \mathcal{C}\mathrm{Sp}^{\Sigma}_{A//A}$$

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There are Quillen adjunctions

$$\mathrm{Mod}_A \underset{}{\overset{Q_A}{\leftrightarrows}} \mathrm{Nuca}_A \underset{I_A}{\rightleftarrows} \mathcal{C}\mathrm{Sp}^{\Sigma}_{A//A}$$

which become Quillen equivalences after stabilization.

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Proposition

There is a natural weak equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A}(\mathrm{TAQ}^R(A), X) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{Sp}^{\Sigma}_{R//A}}(A, A \vee X),$$

and the right-hand side is by definition the space of derivations $\mathrm{Der}_R(A, X)$.

Étale morphisms of ring spectra

A map $R \to A$ of commutative ring spectra is *étale* if $\pi_0(R) \to \pi_0(A)$ is étale and $\pi_0(A) \otimes_{\pi_0(R)} \pi_*(R) \to \pi_*(A)$ is an isomorphism.

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Theorem (Lurie)

Let R be an $\mathbb{E}_\infty$-ring. The functor $\pi_0(-)$ induces an equivalence $\mathrm{CAlg}^{ét}_{R/} \to \mathrm{CRing}^{ét}_{\pi_0(R)/}$ between the category of étale R-algebras to the (nerve of the) category of étale algebras over $\pi_0(R)$.

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- This notion of étaleness is particularly well-behaved between maps of *connective* ring spectra.
- The map $\mathrm{KO} \to \mathrm{KU}$ fails to be étale, despite enjoying many of the formal properties of étale maps.

Formal étaleness properties for HH

Let $R \to A$ be an étale morphism of discrete commutative rings. Weibel-Geller show that $A \otimes_R \mathrm{HH}(R) \xrightarrow{\simeq} \mathrm{HH}(A)$ in this case.

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- Since $\pi_1 \mathrm{HH}^R(A) \cong \Omega^1_{A|R}$, this in turn implies that $\Omega^1_{A|R} \cong 0$.

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- If $\Omega^1_{A|R} \cong 0$, then $R \to A$ is étale as soon as it is flat and finitely presented.

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Let $R \xrightarrow{f} A$ be a map of commutative symmetric ring spectra. We say that f

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Conclusion of forthcoming discussion: downwards implications always hold, upwards under connectivity (and finiteness) hypotheses.

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Proof sketch.

Lurie shows that $R \to A$ étale implies that, for any C,

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(A, C) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(\pi_0(A), \pi_0(C)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(R, C) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(\pi_0(R), \pi_0(C)) \end{array}$$

is cartesian. Mathew deduces $A \wedge_R (S^1 \otimes R) \xrightarrow{\simeq} S^1 \otimes A$ from this. □

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Formal étaleness properties for THH

Example

Consider $\mathrm{KO} \to \mathrm{KU}$. Then

$$\mathrm{THH}(\mathrm{KO}) \simeq \mathrm{KO} \vee \Sigma \mathrm{KO}_{\mathbb{Q}}, \quad \mathrm{THH}(\mathrm{KU}) \simeq \mathrm{KU} \vee \Sigma \mathrm{KU}_{\mathbb{Q}},$$

and it is the case that $\mathrm{KU} \wedge_{\mathrm{KO}} \mathrm{THH}(\mathrm{KO}) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{KU})$.

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$$\mathrm{KU} \xrightarrow{\simeq} \mathrm{THH}^{\mathrm{KO}}(\mathrm{KU}) \text{ and } \mathrm{TAQ}^{\mathrm{KO}}(\mathrm{KU}) \simeq *$$

Similarly for $\mathrm{L}_p \to \mathrm{KU}_p$.

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Similarly for $\mathrm{L}_p \to \mathrm{KU}_p$.

In general, $\mathrm{TAQ}^R(A) \simeq *$ does not imply $A \xrightarrow{\simeq} \mathrm{THH}^R(A)$, which in turn does not imply $A \wedge_R \mathrm{THH}(R) \xrightarrow{\simeq} \mathrm{THH}(A)$.

Formal étaleness properties for THH

Lurie shows that $\mathbb{L}_{A|R} = \mathrm{TAQ}^R(A) \simeq *$ implies étale for R and A connective and $\pi_0(A)$ is finitely presented over $\pi_0(R)$.

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Theorem (Mathew)

Let $R \to A$ is a map of connective commutative symmetric ring spectra with $\mathrm{TAQ}^R(A) \simeq *$. Then $A \wedge_R \mathrm{THH}(R) \xrightarrow{\simeq} \mathrm{THH}(A)$.

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Mathew shows that it is enough to show that

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(A, C) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(\pi_0(A), \pi_0(C)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(R, C) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(\pi_0(R), \pi_0(C)) \end{array}$$

is homotopy cartesian.

Formal étaleness properties for THH

Lurie shows that $\mathbb{L}_{A|R} = \mathrm{TAQ}^R(A) \simeq *$ implies étale for R and A connective and $\pi_0(A)$ is finitely presented over $\pi_0(R)$.

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Formal étaleness properties for THH

In conclusion: there are always downwards implications, while the converse statements hold if R and A are connective:

- étale descent: $A \wedge_R \mathrm{THH}(R) \xrightarrow{\simeq} \mathrm{THH}(A)$.
- formally THH-étale: $A \xrightarrow{\simeq} \mathrm{THH}^R(A)$.
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Non-examples include the connective covers of our examples: the maps

$$\mathrm{ko} \to \mathrm{ku}, \qquad \ell_p \to \mathrm{ku}_p$$

do not satisfy any of these properties.

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Logarithmic derivations

For a map of pre-log rings $(R, P) \to (A, M)$ and A-module X, the *log derivations* sit in the pullback square

$$\begin{array}{ccc} \mathrm{Der}_{(R,P)}((A, M), X) & \longrightarrow & \mathrm{CMon}_{P//M}(M, M \times X) \\ \downarrow & & \downarrow \\ \mathrm{CRing}_{R//A}(A, A \oplus X) & \longrightarrow & \mathrm{CRing}_{\mathbb{Z}[P]//A}(\mathbb{Z}[M], A \oplus X). \end{array}$$

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These are corepresented by $\Omega^1_{(A,M)|(R,P)}$, which contains logarithmic differentials satisfying $\alpha(m)\mathrm{dlog}(m) = d(\alpha(m))$.

Example

If $(R, \langle \pi_R \rangle) \to (A, \langle \pi_A \rangle)$ is a finite extension of DVRs, then $\Omega^1_{(A,\langle \pi_A \rangle)|(R,\langle \pi_R \rangle)} \cong 0$ precisely when $R \to A$ is tamely ramified.

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Equip $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ with positive projective model structure: weak equivalences are checked on Bousfield-Kan homotopy colimits. The category of $\mathcal{J}$-spaces is *not* pointed.

Commutative $\mathcal{J}$-space monoids and symmetric ring spectra

Theorem (Sagave-Schlichtkrull)

There is a chain of Quillen equivalences

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- There is an inclusion $\mathrm{GL}_1^{\mathcal{J}}(A) \subset \Omega^{\mathcal{J}}(A)$ realizing the inclusion of the graded units in $(\pi_*(A), \cdot)$.

Group completion and repletion

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Theorem (Sagave)

The category $\mathcal{CS}^{\mathcal{J}}$ admits a group completion model structure $\mathcal{CS}^{\mathcal{J}}_{\mathrm{gp}}$ in which $M \to N$ is a weak equivalence if and only if $B(M_{h\mathcal{J}}) \to B(N_{h\mathcal{J}})$ is a weak equivalence and the fibrant objects are positive fibrant and grouplike.

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Can often describe N^{rep} as the homotopy pullback of $M \to M^{\mathrm{gp}} \leftarrow N^{\mathrm{gp}}$. For example, $(\mathbb{N} \times \mathbb{N})^{\mathrm{rep}} \cong \mathbb{N} \times \mathbb{Z}$.

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- They provide e.g. a localization sequence $\mathrm{THH}(\tau_{\leq 7}\mathrm{ko}) \to \mathrm{THH}(\mathrm{ko}) \to \mathrm{THH}(\mathrm{ko}, D(w))$.

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This is a situation in which the Bousfield-Friedlander theorem is applicable. □

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Theorem (L.)

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$\mathrm{TAQ}^{(R,P)}(A, M)$ *is naturally weakly equivalent to Sagave's model.*

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- *There is a transitivity sequence*

$$B \wedge_A \mathrm{TAQ}^{(R,P)}(A, M) \to \mathrm{TAQ}^{(R,P)}(B, N) \to \mathrm{TAQ}^{(A,M)}(B, N).$$

- $\mathrm{TAQ}^{(R,P)}(A, M)$ *is logification invariant.*

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- Our results show that, once repletion is taken into account in the model structure, the A-module $\mathrm{TAQ}^{(R,P)}(A, M)$ arises as the A-module underlying the (A, M)-module $\mathbb{L}_{(A,M)|(R,P)}$.

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- satisfies *log étale descent* if $A \wedge_R \mathrm{THH}(R, P) \xrightarrow{\simeq} \mathrm{THH}(A, M)$;
- is *formally log* THH*-étale* if $A \xrightarrow{\simeq} \mathrm{THH}^{(R,P)}(A, M)$;
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Theorem (L.)

There are always downwards implications, and the reverse implications hold as soon as (R, P) and (A, M) are connective.

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Theorem (Sagave)

$(\ell_p, D(v)) \to (\mathrm{ku}_p, D(u))$ is formally log TAQ-étale: $\mathrm{TAQ}^{(\ell_p, D(v))}(\mathrm{ku}_p, D(u))$ is contractible.

Consequences

Corollary

$\mathrm{ku}_p \wedge_{\ell_p} \mathrm{THH}(\ell_p, D(v)) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{ku}_p, D(u))$ *and* $\mathrm{ku}_p \xrightarrow{\simeq} \mathrm{THH}^{(\ell_p, D(v))}(\mathrm{ku}_p, D(u))$.

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- What about $(\mathrm{ko}, D(w)) \to (\mathrm{ku}, D(u))$? How should results about its log THH be interpreted?

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Technical key:
$B^{\mathrm{cy}}_P(M)^{\mathrm{rep}} \cong (S^1 \odot_M (M \boxtimes_P M))^{\mathrm{rep}} \simeq S^1 \odot_M (M \boxtimes_P M)^{\mathrm{rep}}$.

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- There is a conditionally convergent spectral sequence with

$$E_1^{s,t} = \pi_{t-s}(\big[\bigwedge_A^s \Sigma\mathrm{TAQ}^{(R,P)}(A, M)\big]_{h\Sigma_s})$$

converging to $\pi_{t-s}(\mathrm{THH}^{(R,P)}(A, M))$, in analogy with results of Quillen and Minasian.