Postnikov Towers of Logarithmic Ring Spectra

Tommy Lundemo

Logarithmic geometry [7] is a variant of algebraic geometry in which the notions of étaleness and smoothness are less rigid than usual. For example, tamely ramified extensions of (complete) discrete valuation rings (in mixed characteristic with perfect residue fields) participate in log étale morphisms, despite not being étale.

When advertised to homotopy theorists, log structures are often described as "intermediate localizations." By definition, a *(pre-)log ring* (R, P, β) consists of a commutative ring R, a commutative monoid P, and a map $\beta\colon P \to (R, \cdot)$ of commutative monoids. If R is a discrete valuation ring, a choice of uniformizer π_R gives rise to a log structure $\langle \pi_R \rangle \to (R, \cdot)$ on R, simply by including the multiplicative monoid $\langle \pi_R \rangle := \{\pi_R^0, \pi_R^1, \dots\}$ in R. We think of the log ring $(R, \langle \pi_R \rangle)$ as an intermediate localization in-between R and the fraction field $F := R[1/\pi_R]$.

Logarithmic THH**.** The perspective of log structures as "intermediate localizations" is reinforced by THH-cofiber sequences constructed by Rognes–Sagave–Schlichtkrull [13]. For example, dévissage implies that there is a fiber sequence

$$K(\mathbb{F}_p) \to K(\mathbb{Z}) \to K(\mathbb{Z}[1/p])$$

in algebraic K-theory. This does not work for THH: One cannot identify the fiber of $\mathrm{THH}(\mathbb{Z}) \to \mathrm{THH}(\mathbb{Z}[1/p])$ with $\mathrm{THH}(\mathbb{F}_p)$. The introduction of [4] highlights this point very eloquently.

One can associate to any log ring (R, P) a commutative R-algebra in spectra $\mathrm{THH}(R, P)$ [12, Definition 8.11]. It is shown in [13, Theorem 5.5, Example 5.7] that this construction participates in a cofiber sequence

$$\mathrm{THH}(R) \to \mathrm{THH}(R, \langle x \rangle) \to \mathrm{THH}(R/x)[1]$$

for any non-zero divisor x in R. The available constructions of the cofiber sequence are *not* an instance of dévissage but rather come to life by a direct analysis of the map $\mathrm{THH}(R) \to \mathrm{THH}(R, \langle x \rangle)$. In particular, the construction of the cofiber sequences makes no reference to Morita-invariance type properties of logarithmic THH (at the time of writing, no such property is known to the author).

Consequently, the relationship between logarithmic THH and algebraic K-theory is not at all clear. Nonetheless, the more flexible notion of étaleness in log geometry is useful in this context. For example, there is a base-change formula

$$A \otimes_R \mathrm{THH}(R) \xrightarrow{\simeq} \mathrm{THH}(A)$$

for étale morphisms of commutative rings $R \to A$ (by e.g. [11, Theorem 1.3]). Many examples of log étale morphisms $(R, P) \to (A, M)$ give rise to a base-change formula

$$A \otimes_R \mathrm{THH}(R, P) \xrightarrow{\simeq} \mathrm{THH}(A, M)$$

by (the proof of) [8, Theorem 1.11]; this covers the example of tamely ramified extensions of discrete valuation rings. The analogous property for *log ring spectra* has proven useful for both THH and K-theory computations, as we explain below.

Logarithmic ring spectra. Rognes [12] initiated the study of log structures in the context of structured ring spectra. The role of commutative rings is now played by $\mathbb{E}_\infty$-rings, while that of the monoid is often played by the "QS^0-graded $\mathbb{E}_\infty$-spaces" (or *commutative $\mathcal{J}$-space monoids*) of Sagave–Schlichtkrull [16]. These categories participate in an adjunction which we will denote by

$$\mathbb{S}^{\mathcal{J}}[-] \colon \mathbb{E}_\infty\text{-Spaces}_{/QS^0} \leftrightarrows \mathrm{CAlg}(\mathrm{Sp}) \colon \Omega^{\mathcal{J}}(-),$$

and a *(pre-)log ring spectrum* (R, P, β) is thus an $\mathbb{E}_\infty$-ring R, a QS^0-graded $\mathbb{E}_\infty$-space P, and a map $\beta \colon P \to \Omega^{\mathcal{J}}(P)$. There are important variations of this definition: Replacing QS^0 by $\mathrm{BO} \times \mathbb{Z}$ plays a role in Sagave–Schlichtkrull's [17].

In the present setup, well-behaved log structures $(R, \langle x \rangle)$ arise from homotopy classes $x \in \pi_d(R)$ that are "strict" in a certain sense: We refer to [15, Construction 4.2] for the concrete construction. Examples include connective covers of periodic ring spectra with log structures generated by their periodicity classes; e.g. the connective Adams summand $(\ell_p, \langle v_1 \rangle)$, connective complex K-theory $(\mathrm{ku}, \langle u \rangle)$, and connective real K-theory $(\mathrm{ko}, \langle \beta \rangle)$.

To a log ring spectrum (R, P), Rognes [12, Definition 8.11] and Rognes–Sagave–Schlichtkrull [13, Definition 4.6] associate a commutative R-algebra in spectra $\mathrm{THH}(R, P)$. In the examples $(R, \langle x \rangle)$ of interest, this construction participates in cofiber sequences

$$\mathrm{THH}(R) \to \mathrm{THH}(R, \langle x \rangle) \to \mathrm{THH}(R /\!\!/ x)[1]$$

by [13, Theorem 1.1]. While $\ell_p /\!\!/ v_1 \simeq \mathbb{Z}_p$ and $\mathrm{ku} /\!\!/ u \simeq \mathbb{Z}$, we have $\mathrm{ko} /\!\!/ \beta \simeq \tau_{\leq 7}\mathrm{ko}$, which highlights the lack of reliance on dévissage in the construction of the cofiber sequences in logarithmic THH. Related to this point are the cofiber sequences

$$\mathrm{THH}(\mathrm{BP}\langle n \rangle) \to \mathrm{THH}(\mathrm{BP}\langle n \rangle, \langle v_n \rangle) \to \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)[1]$$

obtained from a corresponding sequence for MUP ([17, Example 8.6]) by using the $\mathrm{MU}[x]$-algebra stuctures on $\mathrm{BP}\langle n \rangle$ of Hahn–Wilson [6], as sketched in e.g. [5, Remark 9.8]. Results of Barwick–Lawson [2] and Antieau–Barthel–Gepner [1] suggest that this is an apparent mismatch with the corresponding sequences in algebraic K-theory, which adds to the difficulty of giving K-theoretic interpretations of the cofiber sequences in logarithmic THH.

Logarithmic deformation theory. To any map of log ring spectra one can associate a *log cotangent complex* $\mathbb{L}_{(A,M)/(R,P)}$ [12, 15, 8]. Analogously to the situation for ordinary THH (cf. the argument of [11]), its vanishing implies base-change

$$A \otimes_R \mathrm{THH}(R, P) \xrightarrow{\simeq} \mathrm{THH}(A, M)$$

in logarithmic THH in the connective case [8, Theorem 1.7]. By [15, Theorem 1.6], the log cotangent complex associated to the inclusion of the Adams summand vanishes, and so we obtain that

$$\mathrm{ku}_p \otimes_{\ell_p} \mathrm{THH}(\ell_p, \langle v_1 \rangle) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{ku}_p, \langle u \rangle);$$

this is also the content of [14, Theorem 1.5]. This is computationally useful in conjunction with the cofiber sequences in logarithmic THH: In [14], this is used to recover Ausoni's computation of $V(1)_*\mathrm{THH}(\mathrm{ku}_p)$, while Bayındır [3] has used these methods to recover Ausoni's computation of $T(2)_*K(\mathrm{ku}_p)$ in terms of $T(2)_*K(\ell_p)$.

The presence of a cotangent complex and the more flexible notion of étaleness in log geometry naturally begs the question of an obstruction theory with vanishing obstruction groups for log étale extensions. As a first step, we would like to understand the logarithmic analog of the tower of square-zero extensions

$$\cdots \to \tau_{\leq 2}(R) \to \tau_{\leq 1}(R) \to \tau_{\leq 0}(R) \simeq \pi_0(R)$$

for a connective ring spectrum R, and how to set up an inductive lifting procedure starting from a formally étale map out of its bottom-most stage.

One should first understand the analog of square-zero extensions for log ring spectra. At this point, there is some tension between

(1) the natural guess from a log geometric perspective, where a square-zero extension $(\widetilde{R}, \widetilde{P}) \to (R, P)$ is one of underlying commutative rings that is *strict*; for the purposes of this exposition, one may read this as $\widetilde{P} \simeq P$.[1]
(2) the natural guess from the perspective of derived/higher algebra, where one would ask that a square-zero extension $(\widetilde{R}, \widetilde{P}) \to (R, P)$ is pulled back from a "log derivation" $(d, d^\flat)\colon (R, P) \to (R \oplus J[1], P \oplus J[1])$; these are corepresented by the log cotangent complex.

Theorem. *These two notions of log square-zero extensions agree.*

That (2) implies (1) appears in [9, Chapter 4], while the converse is currently being written up. For a log ring spectrum (R, P), this gives rise to an essentially unique tower

$$\cdots \to (\tau_{\leq 2}(R), P) \to (\tau_{\leq 1}(R), P) \to (\pi_0(R), P)$$

of log square-zero extensions compatible with the Postinkov tower. This is quite natural from a log geometric perspective: For instance, the "residue field" associated to the log ring $(A, \langle \pi_A \rangle)$ for a discrete valuation ring A is $(A/\pi_A, \langle \pi_A \rangle)$, where all positive powers of π_A map to zero. This is the *standard log point*, of which we consider the log ring spectrum $(\pi_0(R), P)$ to be an analog.

For example, if $(R, P) = (\ell_p, \langle v_1 \rangle)$, let us write $\langle \sqrt[p-1]{v_1} \rangle$ for the object called E in [15, Proof of Prop 4.15]. Then $(\mathbb{Z}_p, \langle v_1 \rangle) \to (\mathbb{Z}_p \otimes_{\mathbb{S}^{\mathcal{J}}[\langle v_1 \rangle]} \mathbb{S}^{\mathcal{J}}[\langle \sqrt[p-1]{v_1} \rangle], \langle \sqrt[p-1]{v_1} \rangle)$ is formally log étale, and the underlying ring spectrum of the target is equivalent to $\mathbb{Z}_p \otimes_{\ell_p} \mathrm{ku}_p$. Formally log étale maps lift uniquely along log square-zero extensions [9, Theorem 4.1.0.3]. We are currently pursuing more structured statements relating the categories formally log étale of (R, P)- and $(\pi_0(R), P)$-algebras. For this, we extend Lurie's cotangent complex formalism [10, Section 7.3] to the context of log geometry: The expected identification of the fibers of the resulting *replete tangent bundle* $T^{\mathrm{rep}}_{\mathrm{Log}}$ is available in [9, Proposition 5.1.0.1].

[1] Making this precise would require making the distinction and passage between pre-log and log ring spectra explicit. The definition appears in e.g. [12, Definition 7.25].

References

[1] B. Antieau, T. Barthel, D. Gepner, *On localization sequences in the algebraic K-theory of ring spectra*, J. Eur. Math. Soc. (JEMS) **20** (2018), no.2, 459–487.

[2] C. Barwick, T. Lawson, *Regularity of structured ring spectra and localization in K-theory*, arXiv:1402.6038

[3] H. Bayındır, *Algebraic K-theory of the two-periodic first Morava K-theory*, arXiv:2305.14308

[4] R. Burklund, I. Levy, *On the K-theory of regular coconnective rings*, Selecta Math. (N.S.) **29** (2023), no.2, Paper No. 28, 30 pp.

[5] F. Binda, T. Lundemo, D. Park, P.A. Østvær, *A Hochschild-Kostant-Rosenberg Theorem and Residue Sequences for Logarithmic Hochschild Homology*, arXiv:2209.14182

[6] J. Hahn, D. Wilson, *Redshift and multiplication for truncated Brown-Peterson spectra*, Ann. of Math. (2) **196** (2022), no.3, 1277–1351.

[7] K. Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224.

[8] T. Lundemo, *On the relationship between logarithmic TAQ and logarithmic THH*, Doc. Math. **26** (2021), 1187–1236.

[9] T. Lundemo, *On Formally Étale Morphisms in Derived and Higher Logarithmic Geometry*, PhD Thesis, Radboud University Nijmegen (2022)

[10] J. Lurie, *Higher Algebra*, available on the website of the author.

[11] A. Mathew, *THH and base-change for Galois extensions of ring spectra*, Algebr. Geom. Topol. **17** (2017), no.2, 693–704.

[12] J. Rognes, *Topological Logarithmic Structures*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), 401–544. Geom. Topol. Monogr., 16.

[13] J. Rognes, S. Sagave, C. Schlichtkrull, *Localization sequences for logarithmic topological Hochschild homology*, Math. Ann. **363** (2015), no.3-4, 1349–1398.

[14] J. Rognes, S. Sagave, C. Schlichtkrull, *Logarithmic topological Hochschild homology of topological K-theory spectra*, J. Eur. Math. Soc. (JEMS) **20** (2018), no.2, 489–527.

[15] S. Sagave, *Logarithmic structures on topological K-theory spectra*, Geom. Topol. **18** (2014), no.1, 447–490.

[16] S. Sagave, C. Schlichtkrull, *Diagram spaces and symmetric spectra*, Adv. Math. **231** (2012), no.3-4, 2116–2193.

[17] S. Sagave, C. Schlichtkrull, *Virtual vector bundles and graded Thom spectra*, Math. Z. **292** (2019), no.3-4, 975–1016.